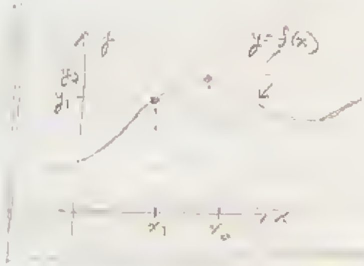
Complex Differentiation (Cauchy-Riemann Equations)

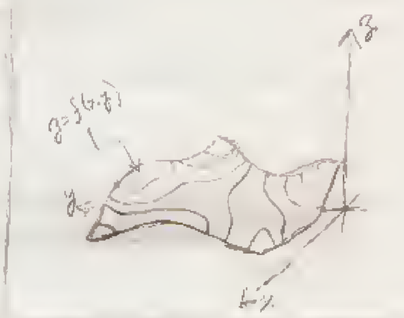
Let's [illegible] by [illegible] and [illegible] of complex variable. [illegible] starting with the [illegible] look at [illegible] functions of real [illegible]..



$f : \mathbb{R} \to \mathbb{R}$

For each real value (in the domain of f), the function yields a real value (in the range of f). [illegible]

$f : \mathbb{R}^2 \to \mathbb{R}$



This is a function that maps a pair of real values to a single real value. All together, these real values form a surface over the 2-dimensional domain of f.

$f : \mathbb{C} \to \mathbb{R}$

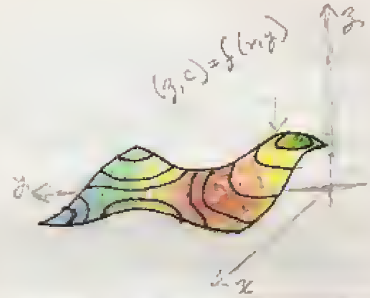
This is exactly the same as $\mathbb{R}^2 \to \mathbb{R}$, because a complex value can [illegible] a pair of real values:

$$z = x + iy \; ; \quad z \in \mathbb{C}; \; x \in \mathbb{R}, \; y \in \mathbb{R}$$

So this function is also a surface, over the complex plane.

→

$f: \mathbb{R}^2 \to \mathbb{R}^2$



Now f is a function of two real values, whose value is also a pair of real values. It's a bit hard to contain the geometric visualization of the points the function produces a surface over a domain, but the [illegible] some [illegible] completely independent property over the plane. In the figure, we use color to represent the [illegible] of the function [illegible]. Notice [illegible] [illegible] related [illegible] the [illegible] the surface: the [illegible] are [illegible] parallel.

$f: \mathbb{C} \to \mathbb{C}$

Once again this is mostly the same as $\mathbb{R}^2 \to \mathbb{R}^2$ except that the [illegible] is different (as is the nature of complex values).

Differentiation (Cauchy-Riemann Equations)

[illegible] could produce [illegible] variable $f: \mathbb{R} \to \mathbb{R}$, [illegible] the [illegible] function [illegible] [illegible] [illegible] we keep [illegible] [illegible] [illegible] the limit of this process, [illegible] connecting the two points [illegible] [illegible] [illegible] of the function at the [illegible].

We do the same for the [illegible] of a surface. Take two points on the surface & [illegible] & [illegible] move them [illegible] closer. We [illegible] [illegible] [illegible] syntax

$$f'(x,y) = \lim_{(a,b)\to(x,y)} \frac{f(a,b) - f(x,y)}{(a,b) - (x,y)}$$

The thing is, for this to be differentiable at the point (x,y), the limit needs to exist & be the same no matter where you approach from. In other words, no matter where a & b start from in the above limit, the limit must be the same. [illegible] like we have for differentiability $\mathbb{R}\to\mathbb{R}$ vectors ([illegible] [illegible] [illegible] $|x|$ is not differentiable).

So now let's [illegible] work (for $\mathbb{C}$) to $\mathbb{C}\to\mathbb{C}$, & evaluate the derivative at a point.

First, assume that $f(x+iy) = u(x+iy) + iv(x+iy)$ where $u(\cdot)$ & $v(\cdot)$ are each real valued functions of complex variables ($u: \mathbb{C}\to\mathbb{R}$, $v: \mathbb{C}\to\mathbb{R}$).

We'll start by evaluating the [illegible] of [illegible] as we approach along a path parallel to the real axis, [illegible]..

$$f'(x+iy) = \lim_{h\to 0} \frac{f(x+h+iy) - f(x+iy)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h+iy) + iv(x+h+iy) - u(x+iy) - iv(x+iy)}{h}$$

(cont)

$$f'(x+iy) = \lim_{h\to 0} \frac{u(x+h+iy) - u(x+iy)}{h} + i \lim_{h\to 0} \frac{v(x+h+iy) - v(x+iy)}{h}$$

$$\therefore f'(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Now, when we evaluate the limit as we approach along a path parallel to the imaginary (y) axis, we get since f is differentiable:

$$f'(x+iy) = \lim_{h\to 0} \frac{f(x+i(y+h)) - f(x+iy)}{ih}$$

$$= \lim_{h\to 0} \frac{u(x+i(y+h)) + iv(x+i(y+h)) - u(x+iy) - iv(x+iy)}{ih}$$

$$= \lim_{h\to 0} \frac{u(x+i(y+h)) - u(x+iy)}{ih} + i \lim_{h\to 0} \frac{v(x+i(y+h)) - v(x+iy)}{ih}$$

$$\frac{1}{i} \lim_{h\to 0} \frac{u(x+i(y+h)) - u(x+iy)}{h} + \frac{i}{i} \lim_{h\to 0} \frac{v(x+i(y+h)) - v(x+iy)}{h}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\therefore f'(x+iy) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad \left(\text{because } \frac{1}{i} = -i\right)$$

So we have two expressions for $f'(x+iy)$, which must be equal:

$$f'(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Setting the real & imaginary parts equal to each other we get the <u>Cauchy-Riemann</u> equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

And the point of all this is that for a function to be complex differentiable, it must satisfy the Cauchy Riemann Equations

This space intentionally
left blank.